Minimal Heegaard splitting and other results concerning 3-manifold.

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SUMMARY

In Chapter I we study the equivalence of boundary incompressibility and homotopy boundary incompressibility of proper surfaces in a three-manifold.

In Chapter II we study Heegaard splittings of an irreducible three-manifold and additivity of Heegaard genus.

In Chapter III we characterize a Heegaard diagram by a family of loops not necessarily simple or disjoint.

CONTENTS

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On Boundary compressible surfaces.

Chapter I.

§1. Introduction.

In this chapter we will show the equivalence of boundary incompressibility and homotopy boundary incompressibility (see definitions 1 and 2) in the case of a proper two-sided incompressible surface in a 3-manifold. We will also construct an example to show that the hypothesis that the surface be two-sided is essential.

I am indebted to David B.A. Epstein for useful discussions, and specially for having proposed this problem.

§2. Definitions and lemmas.

All maps and spaces will be piecewise linear in this chapter.

Definition 1.

Let S be a proper surface in a 3-manifold M (i.e. $S \cap \partial M = \partial S$) . S is said to be *boundary compressible* in M (∂-compressible) if there exists an embedded disc D in M such that:

i) $D \cap S = \Gamma_1$, $D \cap \partial M = \Gamma_2$ where Γ_1 is an arc in ∂D and $\Gamma_2 = \mathrm{Cl}(\partial D \setminus \Gamma_1)$.

ii) Γ_1 is an essential arc in S (i.e. Γ_1 is not homotopic in S into ∂S rel $\partial \Gamma_1$).

If S is not ∂-compressible we say S is ∂-incompressible.

Definition 2.

Let S be a proper surface in a 3-manifold M. S is said to be *homotopy boundary compressible* in M if there exists a proper (possibly singular) arc γ in S (i.e. $\partial\gamma \subset \partial S$) such that

i) γ is homotopic in M into ∂M rel $\partial\gamma$.

ii) γ is an essential arc in S.

If S is not homotopy ∂-compressible we say S is homotopy ∂-incompressible.(*)

Note that in definition 2 we have not only allowed the disc D of definition 1 to be a singular disc, but we have also allowed $D \setminus \partial D$ to meet S.

If M is a manifold we can construct a new manifold dM (the double of M) by glueing together, by the identity map the boundaries of two copies of M. dM is characterized by

i) $M \subset dM$ and ii) there exists a homeomorphism $R: dM \to dM$ such that $R^2 = id$, $R|_{\partial M} = id$, $dM = M \cup R(M)$ and $\partial M = M \cap R(M)$.

If K is any set in M, we write $dK = K \cup R(K)$. For example, if γ is a proper arc in M then $d\gamma$ is a loop in dM.

(*) Another way to define it is:

S is homotopy ∂-incompressible if and only if $\pi(S, \partial S) \hookrightarrow \pi(M, \partial M)$ is injective. Here, and throughout the thesis $\pi = \pi_1$.

Lemma 1.

Let γ be a proper arc in a surface S . Then γ is an essential arc in S if and only if $d\gamma$ is a non-trivial loop in dS .

Proof.

i) Suppose $d\gamma \simeq 1$ in dS and let $\Psi : B \to dS$ be the homotopy where B is a disc. Without loss of generality we can suppose Ψ is transverse to $\partial S \subset dS$. Then $\Psi^{-1}(\partial S)$ is a union of 1-spheres S^1_j , $j = 1,\dots,p$ and a single arc L .

L divides B into two discs B_1 and B_2 . If there is no S^1_j in $\psi^{-1}(\partial S)$ then $(\Psi|_{B_1})$ or $(\psi|_{B_2})$ gives us the desired homotopy of γ into ∂S rel $\partial\gamma$. So we are going to construct a new homotopy $\tilde{\psi} : B \to dS$ with fewer 1-spheres in $\tilde{\psi}^{-1}(\partial S)$ and the result will follow by induction.

Let $S^1_j \subset \psi^{-1}(\partial S)$ be an innermost 1-sphere in B . Then S^1_j bounds a disc $B^1 \subset B$. Now define $\bar{\psi} : B \to dS$ by

$$\bar{\psi}|_{(B\setminus \mathrm{int}\, B^1)} = \psi|_{(B\setminus \mathrm{int}\, B^1)} \quad \text{and} \quad \bar{\psi}_{B^1} = R\circ\psi|_{B^1} \quad .$$

If we make $\bar{\psi}$ transverse to ∂S in an obvious way we get $\tilde{\psi}$ with fewer 1-spheres in $\tilde{\psi}^{-1}(\partial S)$. //

ii) The other implication is obvious.

Lemma 2.

Let S be a proper surface in a 3-manifold M. Then S is (homotopy) ∂-compressible or compressible in M if and only if $dS \subset dM$ is a (homotopy) compressible surface in dM.

Proof (=>)

If S is a compressible surface in M, so is dS in dM.

If S is ∂-compressible and D is the disc in definition 1, dD gives us the compressible disc for dS in M. By Lemma 1 $\partial(dD)$ is an essential loop in dS.

(The same proof holds if S is homotopy ∂-compressible.)

(<=)

Let L be an essential loop in dS which is trivial in dM by the homotopy $\psi: B \to dM$ where B is a disc. Suppose ψ is transverse to $\partial M \subset DM$. Then $\psi^{-1}(\partial M)$ is a union of disjoint 1-spheres and proper arcs.

a) It is possible to find a new map $\tilde{\psi}: B \to dM$ with $\tilde{\psi}|_{\partial B} = \psi|_{\partial B}$ transverse to ∂M such that there is no 1-sphere in $\tilde{\psi}^{-1}(\partial M)$ and $\psi^{-1}(\partial M)$ has the same number of arcs as $\psi^{-1}(\partial M)$.

Proof

Change Ψ to $\tilde{\Psi}$ as in lemma 1 in a singular case. If Ψ is non singular let $S^1 \subset \Psi^{-1}(\partial M)$ be an innermost 1-sphere in B, which bounds a disc $B^1 \subset B$. Then $\Psi(B^1) \subset M$ or $\Psi(B^1) \subset RM$. Suppose $\Psi(B^1) \subset M$: $\Psi(S^1)$ divides the boundary of a regular neighbourhood of $\Psi(B^1)$ in M into two discs. Let D be one of these discs. Define $\bar{\Psi}$ to be equal to Ψ

in $B\setminus \text{int } B^1$ and extend this map to a homeomorphism of B^1 into RD. If we make Ψ transverse to $\partial M \subset DM$ (rel ∂B), in an obvious way we get $\tilde{\Psi}$ with fewer 1-spheres in $\tilde{\Psi}^{-1}(\partial M)$.

If we put $\tilde{\psi}$ in general position (rel ∂D), the only singularities of $\tilde{\psi}$ are double simple loops which can be removed without introducing new points in $\tilde{\psi}^{-1}(\partial M)^{(*)}$. (See picture 1.) Now a follows by induction.

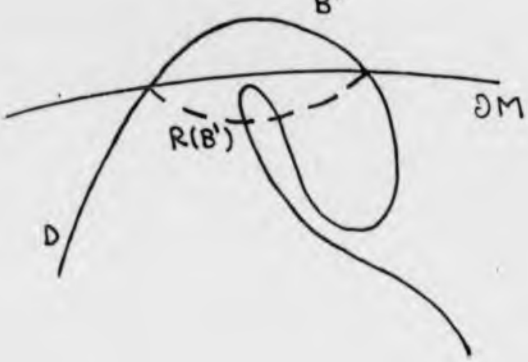
Suppose $\psi^{-1}(\partial M)$ consists entirely of arcs. Let A be an arc in $\psi^{-1}(\partial M)$ and let $\gamma \subset \partial B$ be such that $A \cup \gamma$ bounds a disc $B^1 \subset B$ with $(\text{int } B^1) \cap \psi^{-1}(\partial M) = \emptyset$.

i) If $(\psi|_\gamma)$ is homotopic (in S) into ∂S rel $\partial\gamma$, let $h:D \to S$ be this homotopy and suppose $D \cap B = \gamma$ and $h|_\gamma = \psi|_\gamma$. Now define $\tilde{\psi}:B \cup D \to M$ by $\tilde{\psi}|_B = \psi$ and $\tilde{\psi}|_D = h$. If we make $\tilde{\psi}$ transverse to ∂M in an obvious way, we get a new $\tilde{\psi}$ with one arc less in $\tilde{\psi}^{-1}(\partial M)$ but with a new 1-sphere which can be removed by a). Note that $\psi|_{\partial B}$ is homotopic to $\tilde{\psi}|_{\partial(B\cup D)}$ in dS.

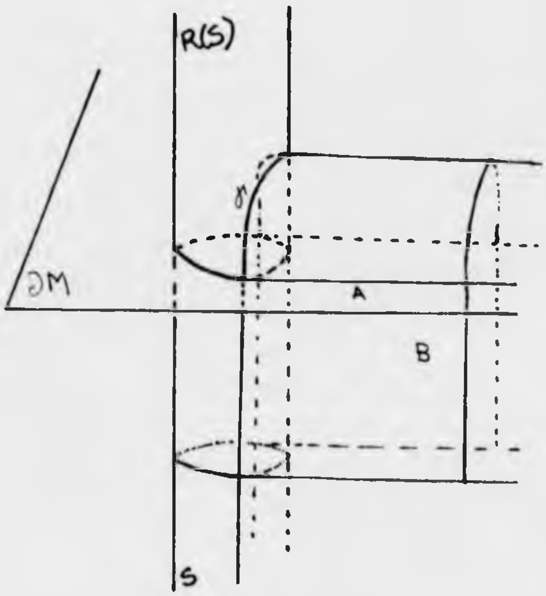
If $\psi:B \to dM$ is nonsingular then $\psi(\gamma)$ separates S into two components, one of which is homeomorphic to a disc (say D). Using this disc define an isotopy of (dM, dS), which is the identity outside a neighbourhood of D and which sends ψ to $\tilde{\psi}:B \to dM$ such that $\tilde{\psi}$ is transverse to ∂M and $\tilde{\psi}^{-1}(\partial M)$ contains fewer arcs and some new 1-spheres which can be removed. (See picture 2 and 3.)

ii) If γ is an essential arc in S our lemma is proved.

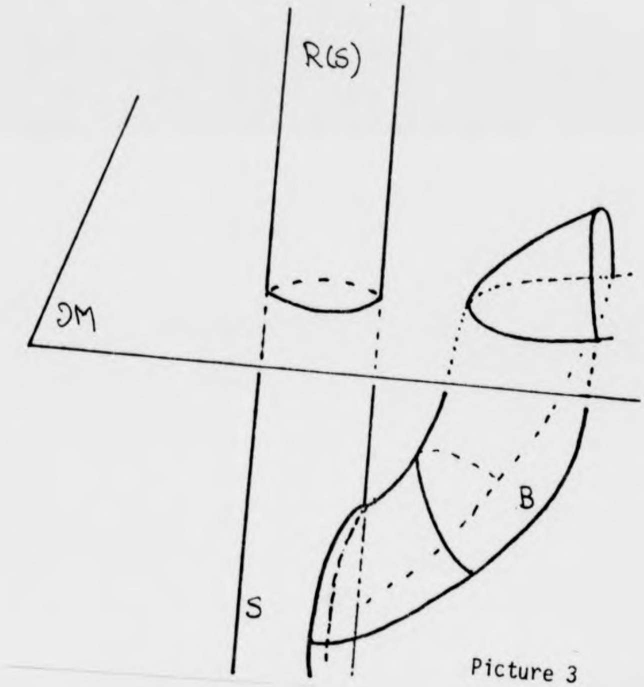
(*) Here we use surgery (see [2], chapter 4.) and the fact that $\Psi(B^1) \subset M$ or $\Psi(B^1) \subset RM$.



Picture 1



Picture 2



Picture 3

If ii) never happens we can find a new map $\tilde{\psi}: B \to dM$ with $\tilde{\psi}|_{\partial B}$ homotopic to $\psi|_{\partial B}$ in dS, such that $\tilde{\psi}(B)$ lies in M or $R(M)$. Therefore S is homotopy compressible in M (or S is compressible in M if ψ is non-singular).

§3. Equivalence and example.

Theorem.

Let S be a two-sided proper incompressible surface in a 3-manifold M. Then S is ∂-compressible if and only if S is homotopy ∂-compressible.

Proof.

Since $dS \subset dM$ is two-sided, by the disc theorem of Papakyriakopoulos [5], dS is compressible in dM if and only if dS is homotopy compressible in dM. Now the result follows by lemma 2.

We now construct a proper 1-sided incompressible surface in a 3-manifold which is homotopy ∂-compressible but not ∂-compressible. (In fact this manifold will be a Haken 3-manifold.)

Let S_0 be a disc with three cross-caps ∂S_0 homeomorphic to a 1-sphere. We embed S_0 in a solid torus V such that ∂S_0 is homologous to $1[m] + 6[\ell]$ where m is a meridian of V, ℓ is some simple loop in ∂V which meets m once and [ ] means the homology class in ∂V. S_0 is constructed by taking $\bar{m}$, a meridian disc of V, as a 0-handle and attaching three 1-handles in $\partial\mathcal{V}$ (see picture 4).

Now if we glue to $\mathcal{V}$ a new solid torus $\mathcal{W}$, identifying ∂V with ∂W so that the meridian u of $\mathcal{W}$ is glued to ∂S, we get the Lens space $L(6,1)$ and $S_1 = S_0 \cup$ (meridian disc of $W) \subset L(6,1)$ is an incompressible surface(*) (see [3] page 211).

Let S^1 be a simple loop in the interior of $\mathcal{W} \subset L(6,1)$ such that:

i) S^1 crosses through the meridian disc $\bar{u}$ of W in a single point;

ii) $\alpha = (S^1 \cap \tilde{W})$ is a knotted arc in $\tilde{W}$ where $\tilde{W}$ is obtained by cutting W along $\bar{u}$.

Let $M = (L(6,1) \setminus$(regular neighbourhood of S^1 in $W)$ and $S = (M \cap S_1)$. Note that S is homeomorphic to S_0.

Now we prove that (M,S) has the required properties.

i) S is a 1-sided incompressible surface in M.

Proof. Suppose there exists a disc D meeting S only in ∂D, with (∂D) an essential loop in S. Since S_1 is an incompressible surface in $L(6,1)$, ∂D is homotopic to ∂S in S. So in $L(6,1)$ we can find a non-separating 2-sphere, which is absurd. //

(*) This is a classical Stalling example of homotopic compressible one-side surface in a three-manifold, which is an incompressible surface.

ii) S is homotopy ∂-compressible

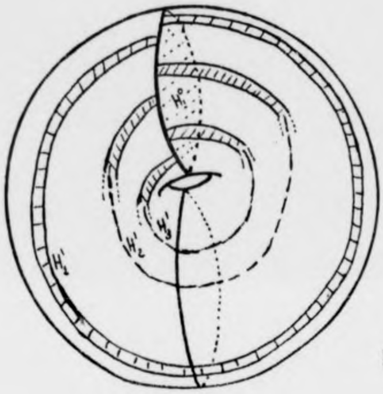
Proof. There is a simple loop L in $S_0 \subset \mathcal{V}$ which is not homotopic to ∂S in S and which is homologous to zero in $\mathcal{V}$ (take L as the boundary of a torus with a hole embedded in S_0). Then L is homotopically trivial in $\mathcal{V}$ and therefore in M. By a homotopy in S, we can suppose that some arc $\Gamma_2 \subset L$ lies in $\partial S \subset \partial M$. Then $\Gamma_1 = Cl(L \setminus \Gamma_2)$ gives us a non-essential arc in M, which is essential in S. //

iii) S is ∂-incompressible.

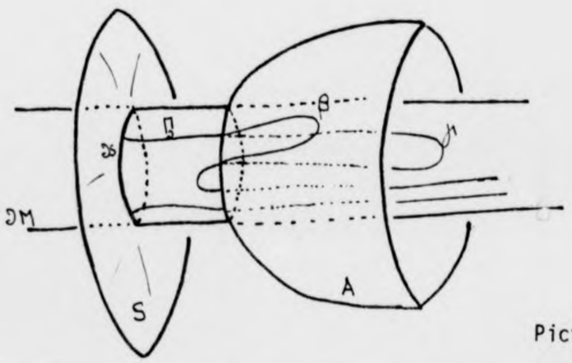
Proof. Suppose not and let D, Γ_1, Γ_2 be as in definition 1. We will show that Γ_2 is isotopic in M (by an isotopy which keeps $\partial\Gamma_2$ fixed) to an arc in ∂S. But this implies that S is a compressible surface in M, contrary to i).

The intersection of D with the annulus $A = (\partial\tilde{W}) \cap M$, after a small isotopy of D, is a disjoint union of 1-spheres and arcs. But since A is an incompressible surface in M (because S is incompressible) without loss of generality we can consider $A \cap D$ to consist only of arcs. There is an innermost arc in D giving us a disc $B \subset D$ meeting A along β and ∂M in $\gamma = Cl(\partial B \setminus \beta) \subset \Gamma_2$.

Since α is a knotted arc in $\bar{W}$, γ is isotopic (by an isotopy in M which keeps S fixed) into a regular neighbourhood of ∂S in ∂M



Picture 4



Picture 5

(see picture 5) and after this isotopy the new disc D meets A in fewer arcs and in one simple loop which can be removed. By induction on the number of arcs in $A \cap D$ we can suppose $A \cap D = \emptyset$. So Γ_2 is isotopic into ∂S (by an isotopy which keeps $\partial\Gamma_2$ fixed) and the proof is complete.

Note 1.

Using the same argument as in iii) we can prove that M is a Haken manifold.

Note 2.

The theorem of this paper still holds if we define a relative form of (homotopy) ∂-compressible. i.e. Let A be a 2-dimensional submanifold (possibly with boundary) of ∂M , which is a closed subspace of ∂M .

Definition 3.

Let S , M , D , Γ_1 , Γ_2 be as in definition 1. Then S is A-∂ compressible if $\Gamma_2 \subset A$. (A-homotopy ∂-compressible if γ in definition 2 can be homotoped in M into $A \subset \partial M$) .

The relative version of the lemma 2 is proved the same way, but changing dM to $d_A M$ where $d_A M$ is the new manifold obtained by taking two copies of M and glueing along $A \subset \partial M$ by the identity map. //

A CONDITION FOR A HEEGAARD SPLITTING OF AN IRREDUCIBLE THREE-MANIFOLD TO BE MINIMAL AND ADDITIVITY OF HEEGAARD GENUS

CHAPTER II

§1. Introduction.

Let M be a connected, closed orientable 3-manifold. A pair (X,X') of handlebodies X and X' is called a Heegaard splitting of M if $M = X \cup X'$ and $X \cap X' = \partial X$. For each Heegaard splitting of M we can find a 1-dimensional simplicial complex Γ in M such that X' is a regular neighbourhood of Γ in M (we write $N(\Gamma, M)$ or just $N(\Gamma)$). Let A be a simple loop in Γ and let $\hat{M} = Cl(M \setminus N(A))$.

In this chapter we will prove that.

Theorem 1.

If M is also an irreducible 3-manifold then, for any Heegaard splitting (X,X') of M either;

i) there exists another Heegaard splitting $(\tilde{X},\tilde{X}')$ of M with genus $\tilde{X}$ < genus X or

ii) $\hat{M}$ (as constructed above) is an irreducible 3-manifold. Furthermore $\partial\hat{M}$ is incompressible in $\hat{M}$ if genus $M > 1$. (*)

Remark 1.

In fact, in theorem 1, if $\hat{M}$ fails to be an irreducible 3-manifold,

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(*) If $\partial\hat{M}$ is compressible, together with the fact that $\partial\hat{M} \cong$ torus and $\hat{M}$ is irreducible, implies that $\hat{M}$ is a solid torus, so genus $M \leq 1$

we can find a pair of meridional discs v of X and w of X' such that $v \cap w = \partial v \cap \partial w$ = a single crossing point and $\tilde{X}$ is constructed by adding to X a small regular neighbourhood of w in X'.

Also, we will prove that the genus of a compact 3-manifold is additive with respect to connected sum, and also with respect to disc sum. This is a generalization of a theorem due to Haken (see [1]), in that we do not restrict our attention to closed 3-manifolds.

In §5 we give a characterization of unknotted arcs in a handlebody.

§2. Definitions.

Let H be a handle decomposition of a compact connected orientable 3-manifold M such that

$$M = \bigcup_{j=1}^{p} H_j^{i_j} \text{ where } H_j^{i} \text{ is an i-handle of } H, \ i_j \leq i_{(j+1)}$$

and $H_j^{i} \cap H_s^{i} = \emptyset$ if $j \neq s$.

Let $X = \bigcup_{i_j \leq 1} H_j^{i_j}$. Then X is a handlebody. The standard definition of Heegaard genus of H is given by:

<u>Definition 1</u> - genus (H) = genus of X. The genus of M (we write genus (M)) is the minimum of genus H as H varies over all handle decompositions of M.

To avoid working with handle decompositions which obscure the geometry, let us make some definitions.

In this chapter Γ or Γ_i, will be a 1-dimension simplicial complex in M with $(\Gamma \cap \partial M) \subset$ ends of Γ, where by an end of Γ we mean a vertex of Γ, which is a vertex of a unique 1-simplex of Γ.

<u>Definition 2</u> - Γ is said to be a trivialization of M if $X = Cl(M \setminus N(\Gamma))$ is a handlebody, where $N(\Gamma)$ is a small regular neighbourhood of Γ in M.

We define:

genus Γ = genus of X and <u>genus (M)</u> is the minimum of genus Γ where Γ is a trivialization of M.

<u>Example</u>.

If H is a handle decomposition of M (as above) it is easy to find a trivialization Γ of M such that $N(\Gamma) = \bigcup_{i \atop j \geqslant 2} H^i_j$. On the other hand for any trivialization Γ it is possible to find a handle decomposition H of M with $N(\Gamma) = \bigcup_{i \atop j \geqslant 2} H^i_j$. So in both cases we have genus Γ = genus (H) and this implies that the two definitions of genus (M) are equal.

Now we can state a stronger version of Theorem 1.

Theorem 1'.

Let Γ be a trivialization of an irreducible 3-manifold M and let A be any sub-complex of Γ with no connected component of A being contractible. Then either there exists $\tilde{\Gamma}$, a trivialization of M with genus $\tilde{\Gamma}$ < genus Γ, or $\tilde{M} = Cl\,(M \setminus N(A))$ is an irreducible 3-manifold.

It is easy to see that Theorem 1' implies theorem 1.

Our other theorem will be:

Theorem 2.

The genus of a 3-manifold is additive with respect to connected sum and also with respect to disc sum.

Remark 2.

If M_1 and M_2 are two oriented 3-manifolds with $\partial M_i \neq \emptyset$, $i = 1, 2$, $M_1 \#_D M_2$ is a new oriented 3-manifold (called the disc sum of M_1 with M_2) obtained by gluing two discs $D_i \subset \partial M_i$, $i = 1,2$, by an orientation reversing homeomorphism of D_1 with D_2 (D_i with the orientation induced from M_i, $i = 1,2$). Up to homeomorphism the disc sum is well defined once we fix the connected component containing D_i in ∂M_i.

In terms of trivializations the equivalent concept to "handle moves" will be as follows.

<u>Definition 3</u> - We say that Γ_1 differs from Γ_2 by an <u>h-move</u> if there exists a disc D in M such that:

i) $Cl\,(\Gamma_i \setminus \Gamma_j) = A_i$ where A_i is a 1-simplex of ∂D $(i \neq j)$

ii) $D \cap ((\Gamma_1 \cup \Gamma_2) \cup \partial M) = \partial D$

<u>Γ is said to be equivalent to</u> Γ' if and only if there exists a sequence $\Gamma = \Gamma_1, \dots, \Gamma_u = \Gamma'$ such that Γ_i differs from Γ_{i+1} by an h-move, $i = 1, \dots, u-1$.

Let $M_i = Cl(M \setminus N(\Gamma_i))$.

<u>Lemma 1</u>.

If Γ_1 is equivalent to Γ_2 then M_1 is homeomorphic to M_2 . Thus Γ_1 is a trivialization of M if and only if Γ_2 is a trivialization of M . Moreover, genus Γ_1 = genus Γ_2 .

<u>Proof</u>.

We need to prove the lemma only when Γ_1 differs from Γ_2 by an h-move.

Let D be the disc in definition 3. It is easy to see that M_i $(i = 1,2)$ collapses to $Cl(M \setminus N(\Gamma_1 \cup D, M))$ so that we have the required homeomorphism.(*) //

(*) See Rouke, C.P., Sanderson, B.J., Introduction to Piecewise-Linear Topology, Springer Study edition, Springer-Verlag 1982, pg.41

Remark 3.

If M in lemma 1 is also a closed 3-manifold then $\Gamma_1 \cup D$ collapses to Γ_i $(i = 1,2)$ so $N(\Gamma_1)$ is isotopic in M to $N(\Gamma_2)$.

§3. Main lemmas.

Lemma 2.

Let D be an embedded proper disc in M and Γ a trivialization of M . Then there exists a new embedded disc D' with $\partial D'$ isotopic to ∂D in ∂M and a new trivialization Γ' of M such that:

i) $D' \cap \Gamma' = \emptyset$

ii) Γ' is equivalent to Γ .

Proof.

Suppose Γ meets D transversely. The proof will be by induction on the number of points in $D \cap \Gamma$.

Let $\hat{D} = D \cap X$ where $X = Cl(M \setminus N(\Gamma))$ and assume that $\hat{D}$ is incompressible in X . (If not we can find a new disc D' with $\partial D' = \partial D$ and $\#(D' \cap \Gamma) < \#(D \cap \Gamma)$.)

Sub-lemma.

Let D be a proper embedded disc in M, which meets a trivialization Γ of M transversely. Then (with the notation above), if $\hat{D}$ is not a disc and is incompressible in X, there exists a new trivialization Γ' of M (equivalent to Γ) with $\#(D' \cap \Gamma') < \#(D \cap \Gamma)$, where D' is isotopic to D (rel ∂D) in M.

Proof

Claim.

There exists an arc A in D with $A \cap \Gamma \subset \partial A$, on arc G in Γ .

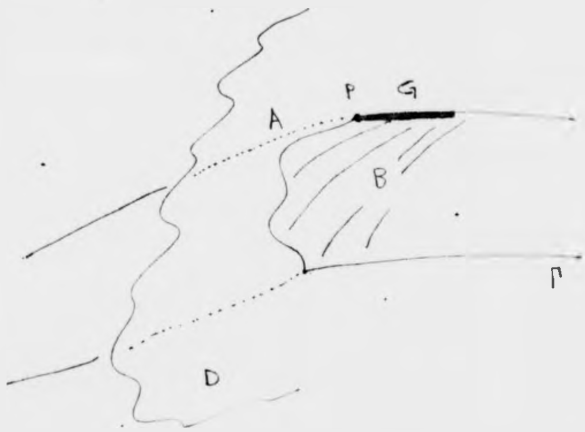
and a (possibly) singular disc(*) B in M such that:

1. $A \cap G = \{p\} = \partial G \cap \partial A$;

2. All singularities of B lie in ∂B but B is non-singular over $\mathring{A} \cup G \subset \partial B$;

3. $B \cap (\Gamma \cup \partial M) = \partial B \setminus \mathring{A}$ (see picture 1).

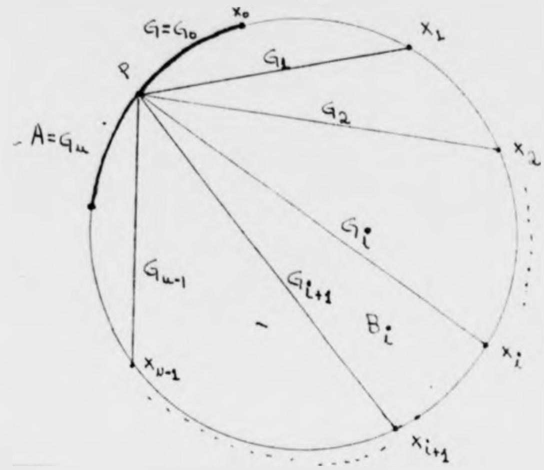
This claim will be proved later.



PICTURE 1.

Now divide ∂B into $x_0, x_1, \ldots, x_u$ (see picture 2) where $x_0 \in \partial G \setminus p$, $x_u \in \partial A \setminus p$ and ∂B is non-singular over the arc $\overline{x_i x_{i+1}}$ $(p \notin \overline{x_i x_{i+1}})$

(*) Here the singular disc B is already in general position.



PICTURE 2.

Let $G_0, G_1, \dots, G_u$ be arcs in B such that $G_0 = G$, $G_u = A$, $\partial G_i = \{p, x_i\}$, G_i is a proper arc in B if $i \neq 0$, u and $G_i \cap G_j = \{p\}$ if $i \neq j$.

Let B_i be the disc in B bounded by $(G_i \cup \overline{x_i x_{i+1}} \cup G_{i+1})$. Let $\Gamma_0 = \Gamma$ and $\Gamma_{i+1} = (\Gamma_i \setminus \mathring{G}_i) \cup G_{i+1}$. Γ_{i+1} differs from Γ_i by an h-move (over the disc B_i). Thus Γ is equivalent to $\Gamma' = (\Gamma \setminus \mathring{G}) \cup A$ and D is isotopic (by a small isotopy) to D' with $\# (D' \cap \Gamma') < \# (D \cap \Gamma)$.

<u>Proof of the claim.</u>

a) There exists a system of meridional discs V of X such that $V \cap \hat{D}$ is a hierarchy for $\hat{D}$. To see this we argue as follows.

Up to isotopy any system of meridional discs V meet $\hat{D}$ along proper arcs or simple loops. Since $\hat{D}$ is incompressible in X the intersection along simple loops can be removed by an isotopy in X.

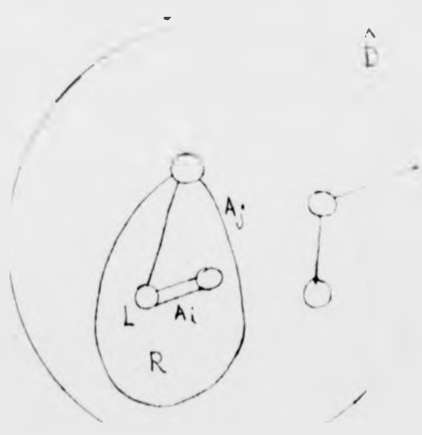
If some arc in $\hat{D} \cap V$ is non-essential we can find a new system of meridional discs which meet $\hat{D}$ in fewer proper arcs than V does.(*)

Since ($\hat{D}$ cut open along $\hat{D} \cap V$) lies in (X cut open along V), which is homeomorphic to a 3-ball, every component of $(\hat{D} \backslash V)$ is a disc, unless $\hat{D}$ is compressible in X. This prove a).

b) Let R be the region in $\hat{D}$ bounded by an innermost separating arc (say A_j) and suppose R does not contain ∂D. If all arcs in $V \cap \hat{D}$ are non-separating, take $R = \hat{D}$ (see picture 3).

c) Let L be any component of ∂D inside R. (Such an L exists because A_j is an essential arc.)

Let $v \in V$ be a disc such that $v \cap L \neq \emptyset$. Among all arcs in $v \cap \hat{D}$ which meet L let A_i be an innermost one in v. Thus there exists a disc $v' \subset v$ such that $\partial v'$ meets $\hat{D}$ along A_i. It is possible that v' meets $\hat{D}$ along other arcs but $\partial v'$ meets L only once.



PICTURE 3.

(*) see foot note page 2.17

Now let $B = v' \cup$ (shadow of $\partial v'$ under the collapse of pairs $(N(\Gamma), N(\Gamma) \cap (D \cup \partial M)) \searrow (\Gamma, \Gamma \cap (D \cup \partial M))$. We define $A = A_i \cup$ {shadow of ∂A_i under the above collapse} .

Since v' meets L once, B is non-singular near some end p of A . So take G to be a small arc in Γ such that $p = G \cap A$, $G \subset \partial B$ and B is non-singular over G . The proves the claim.

Lemma 3.

Let S^2 be an essential two sphere in a 3-manifold M and let Γ be a trivialization of M . Then there exists a new essential sphere $'S^2$ and a new trivialization Γ' of M such that

i) $'S \cap \Gamma'$ is a single crossing point.

ii) Γ' is equivalent to Γ .

Proof.

Suppose S meets Γ transversely. Let $X = \mathrm{Cl}(M \setminus N(\Gamma))$ and $\dot{S} = S \cap X$. If $\dot{S}$ is compressible in X , we can find a new essential sphere $'S$ with $'S \cap X$ incompressible in X and $\#('S \cap \Gamma) < \#(S \cap \Gamma)$. Note that $\#('S \cap \Gamma) \geq 1$ because $'S$ is an essential sphere in M .

If $\hat{S}$ is ~~irreducible~~ INCOMPRESSIBLE in X and $\#(S \cap \Gamma) > 1$. let p be a point in $S \cap \Gamma$ and $\hat{M} = \mathrm{Cl}(M \setminus N(\Gamma))$. We can aply sub-lemma 1 to $D = S \cap \hat{M}$, $\hat{M}$ and $\hat{\Gamma}$ to find a new trivialization Γ' equivalent to Γ , which is transverse to S' with $\#(S' \cap \Gamma') < \#(S \cap \Gamma)$. where S' is isotopic to S in M.

Now the proof will follow by induction.//

§4. Proof of theorems.

Proof of theorem 2.

a) Genus is additive with respect to connected sum.

Use lemma 3 and induction on the number of prime factors of M.(*) //

b) Genus is additive with respect to disc sum.

Let $M = M_1 \#_D M_2$ be the disc sum of M_1 with M_2 and write $M_i = {'M_i} \# {''M_i}$ where $D \supset \partial {'M_i}$ and ${'M_i}$ is an irreducible 3-manifold $(i = 1,2)$. Then

$$\text{I)} \qquad M_1 \#_D M_2 \cong ({'M_1} \#_D {'M_2}) \# ({''M_1} \# {''M_2}) \; .$$

Since ${'M_1} \#_D {'M_2}$ is irreducible D is determined (up to isotopy) by the class of isotopy of ∂D in $\partial({'M_1} \#_D {'M_2})$. By lemma 2 we have:

$$\text{II)} \qquad \text{genus} \, ({'M_1} \# {'M_2}) = \text{genus} \, ({'M_1}) + \text{genus} \, ({'M_2}) \; .$$

Now b) follows if we apply a) and (II) to the right side of (I) . //

Proof of theorem 1'

We first note that if Γ is a trivialization of M then $\hat{\Gamma} = (\Gamma \cap \mathring{M})$ is a trivialization of $\mathring{M}$. Moreover if $\hat{\Gamma}_0$ is equivalent to $\hat{\Gamma}$ in $\mathring{M}$ it is easy to find another trivialization Γ_0 of M equivalent to Γ such that:

(*) Suppose that by induction the genus of a three-manifold is the sum of the genus of its prime factors, provided they are less than n factors, and that M has $n>1$ prime factors in its decomposition. By lemma 3 we can write $M \cong M_1 \# M_2$ $(M_i \neq S^3, i=1,2)$ and genus M = genus M_1 + genus M_2. By uniqueness of decompositions in prime factors the number of prime factors of M_1 and M_2

⟹ cont... footnote page 2.12

$\hat{\Gamma}_0 = (\Gamma_0 \cap \hat{M})$ and $A \subset \Gamma_0$.

Now suppose that $\hat{M}$ is a reducible 3-manifold. Using lemma 3 (for $\hat{M}$ and $\hat{\Gamma} = (\Gamma \cap \hat{M})$) and the remark above we can find an essential sphere S in $\hat{M}$ and a trivialization Γ' of M equivalent to Γ with $A \subset \Gamma'$, such that Γ' meets S in a single crossing point.

Since M is irreducible S bounds a 3-ball B in M which contains at least one component A_i (say) of A in its interior.

Let $\tilde{\Gamma} = Cl(\Gamma' B)$, $\tilde{\Gamma}$ is a trivialization of M , since $Cl(M \setminus N(\tilde{\Gamma})) = Cl(M \setminus N(\Gamma')) \cup B$ is a handlebody because ∂B meets $Cl(M \setminus N(\Gamma'))$ in a disc. Also genus $\tilde{\Gamma} <$ genus $\Gamma' =$ genus Γ since A_i is not contractible.

Proof of theorem 1.

As we saw, theorem 1' implies theorem 1, but we will give a new proof of theorem 1 in order to prove the remark 1. Suppose $\hat{M}$ is a reducible 3-manifold, and Γ is a trivialization of M . As in the proof of theorem 1', we can find a new trivialization Γ' equivalent to Γ with $A \subset \Gamma'$ and a 3-ball B with $A \subset \overset{o}{B}$ such that ∂B is an essential sphere in $\hat{M}$ which meets Γ' in a single crossing point.

Let $\Gamma_1 = \Gamma' \cap B \supset A$. Γ_1 is a trivialization of B with genus $\Gamma_1 \geq 1$; to see this we take

are less than n, and we can use our hypothesis.

This completes our induction.

Using again the uniqueness of prime decomposition, it is easy to see that if $M = M_1 \# M_2$, then genus M = genus M_1 + ... genus M_2.

$$N(\Gamma_1,B) = N(\Gamma') \cap B \text{ ; then } Cl(B\setminus N(\Gamma_1,B)) = Cl(M\setminus N(\Gamma')) \cap B \text{ .}$$

The last factor is a handlebody since $\partial B \cap Cl(M\setminus N(\Gamma'))$ is a disc and $Cl(M\setminus N(\Gamma'))$ is a handlebody.

Let B' be another 3-ball. We glue B to B' along their boundary so as to form a 3-sphere $S^3 = B \cup B'$. Then $(Cl(B\setminus N(\Gamma_1)), N(\Gamma_1) \cup B')$ is a Heegaard splitting of S^3 . By the uniqueness of Heegaard splitting of S^3 (see [7]) we can find a pair of meridional discs $\underline{v}'$ of $Cl(B\setminus N(\Gamma_1))$ and $\underline{w}'$ of $N(\Gamma_1) \cup B'$ with $v' \cap w' = \partial v' \cap \partial w'$ = a single crossing point. Without loss of generality we can suppose that $v' \cap \partial B = w' \cap \partial B = \emptyset$, i.e. v' is a meridional disc of $Cl(M\setminus N(\Gamma'))$ and w' is a meridional disc of $N(\Gamma')$.

Since Γ is equivalent to Γ' , by remark 3, $X' = N(\Gamma)$ is isotopic in M to $N(\Gamma')$ and so $X = Cl(M\setminus N(\Gamma))$ is isotopic in M to $Cl(M\setminus N(\Gamma'))$.

By this isotopy the point corresponding to (u',w') give us the pair (u,w) of meridional discs as described in remark 1. //

Corollary 1.

Let M be a compact closed connected irreducible and orientable 3-manifold then there exists a solid torus that we can drill out of M and we still get an irreducible 3-manifold.

Proof.

If genus $(M) = 0$ this corollary is true $(M \cong S^3)$ so suppose genus $(M) > 0$. Let Γ be a trivialization of M with genus $(M) =$ genus Γ and let A be a simple loop in Γ. Then by theorem 1 we can drill out the solid torus $N(A)$ of M and we still have an irreducible 3-manifold.

Corollary 2. (Bing characterization of S^3)

A connected compact closed 3-manifold M is homeomorphic to S^3 if and only if any simple loop in M lies in a 3-Ball embedded in M.

Proof.

Without loss of generality(*) we can suppose M is an irreducible 3-manifold. Let N be a solid torus such that $Cl(M \setminus N) = \hat{M}$ is an irreducible 3-manifold. By hypothesis N is contained in a 3-ball B. Since $\hat{M}$ is irreducible ∂B bounds a 3-ball in $\hat{M}$ therefore $M \cong S^3$. //

§5. Unknotted arcs in a handlebody.

Lemma 4.

Let Γ and Γ' be proper arcs in a 3-manifold M such that Γ is equivalent to Γ'. Then Γ is isotopic to Γ' in M.

(*) Suppose $M = \#_{i=1}^{n} M_i$, where each of M_i is an irreducible three-manifold. Suppose $\bar{M}_1 = M_1 \cap M$ is homeomorphic to M_1 without the interior of a finite member of a three-ball in M_1.

Let K be a simple loop in $\bar{M}_1 \subset M$. If K is not contained in a three-ball in M_1, then $\hat{M}_1 = Cl\,(M_1 \setminus N(K))$ is an irreducible three-manifold (by irreducibility of M_1), and $\hat{M}_1 \# (\#_{i=2}^{n} M_i)$ is the decomposition of $\hat{M} \cong Cl\,(M \setminus N(K))$ into prime factors. Since K bounds a

⟹ cont.. footnote page 2.15

<u>Proof.</u>

Let $\Gamma_0 = \Gamma, \Gamma_1, \ldots, \Gamma_u = \Gamma'$ be a sequence of h-moves.

a) If Γ_i is not a proper arc then Γ_i must be homeomorphic to $S_i^1 \vee I_i$ where S_i^1 is a 1-sphere in $\overset{o}{M}$ and I_i is an arc in M with $\partial I_i = (I_i \cap S_i^1) \cup (I_i \cap \partial M)$.

Suppose $B_i \times [0,1]$ is embedded in M such that,

i) $B_i \times \{0\}$ is a disc in ∂M and $B_1 \times \{0\} = B_1 \times [0,1] \cap \partial M$;

ii) $I_i = \{P_i\} \times [0,1]$, $P_i \in \overset{o}{B}_i$ and

iii) $S_i^1 \cap B_i \times [0,1]$ is a proper arc A_i in $B_i \times \{1\}$, $\partial A_i = \{c_1, c_2\}$.

Let I_i^1, I_i^2 be two disjoint proper arcs in $\partial B_i \times [0,1]$ with $c_j \in \partial I_i^j$, $j = 1, 2$.

Let $\Gamma_i' = (S_i^1 \setminus A_i) \cup I_i^1 \cup I_i^2$. Note that up to isotopy of M Γ_i' is independent of our choice of the embedding of $B \times [0,1]$ in M and of our choice of I_i^j , $j = 1, 2$.

b) If Γ_i is a proper arc define $\Gamma_i' = \Gamma_i$.

<u>Claim.</u>

If Γ_i differs from Γ_j by an h-move then Γ_i' is isotopic to Γ_j' .

---

three-ball B in M, which by an isotopy of M we can suppose $B \subset M_1$, we have $(\mathrm{Cl}\,(S^3 \setminus N(K))) \# (\#_{i=1}^{n} M_i)$ being another decomposition of M into prime factors, this time with n + 1 factors, which is a contradiction.

Proof.

Since the relation of "differs from" is symmetric we have only 3 - cases to consider.

i) Γ_i and Γ_j are proper arcs in M .

ii) Γ_i is a proper arc but Γ_j is not.

iii) Γ_i and Γ_j are not arcs.

i) follows from the fact that

$$((\Gamma_i \cup D, (\Gamma_i \cup D) \cap \partial M)) \text{ collapses to } (\Gamma_i, \partial\Gamma_i) \text{ or } (\Gamma_j, \partial\Gamma_j)$$

where D is the disc of the h-move of Γ_i to Γ_j . Cases ii) and iii) follow by taking a suitable choice of Γ_i' and Γ_j' in which it is easy to see that Γ_i' differs from Γ_j' by an h-move. //

Theorem 3.

Let X be a handlebody and Γ a proper arc in X . Then Γ is an unknotted arc if and only if $\pi(X\backslash\Gamma)$ is a free group.

Proof.

By induction on genus $X = g$.

If $g = 0$ this is a theorem due to Papakyriakopoulos see [5] .

If $g > 0$, the hypothesis that $\pi(X \setminus \Gamma)$ is a free group implies that Γ is a trivialization of X. So lemmas 2 and 4 allow us to change Γ so as to avoid some meridional disc D of X. Now the result will follow by the inductive hypothesis. //

Another version of this theorem is:

Theorem 4.

Let X be a handlebody of genus g and $X' = X \cup H^2$, where H^2 is a 2-handle. If $\pi(X')$ is a free group on $(g-1)$ generators then X' is a handlebody.

Proof.

Take Γ a proper arc in X' such that $N(\Gamma) = H^2$. Since $\pi(X')$ is free, $\partial X'$ is compressible and lemma 2 gives us the inductive step. For $g = 1$ the theorem is true since there exists no fake 3-ball of genus 1. //

(*) Let B be the disc in $\hat{D}$, determined by an innermost non-essential proper arc of $\hat{D} \cap V$ in $\hat{D}$. If $v \in V$ is the meridian disc which meets $\hat{D}$ in this arc, then the boundary of a regular neighbourhood of $v \cap B$ in X provides us with two new discs v_1 and v_2, which each meets D in fewer proper arcs than v does. Let $'V = \{v_1\} \cup (V \setminus \{v\})$ and $''V = \{v_2\} \cup (V \setminus \{v\})$. $'V$ or $''V$ is a system of meridian discs of X (this is trivial when X is a solid torus and for the general case cut X open along $V \setminus \{v\}$), in the conditions required.

CHARACTERIZATION OF HEEGAARD SPLITTING BY A FAMILY OF LOOPS NOT NECESSARILY SIMPLE OR DISJOINT.

CHAPTER III

§1. Introduction.

Any orientable closed connected 3-manifold M is built by gluing two handlebodies of genus g, X_1 and X_2, along their boundary. This manifold is determined uniquely (up to homeomorphism) if there is given a family of simple disjoint loops $\{v_1,..,v_g\}$ in ∂X_1 such that

i) $$H_1(\partial X_1)/_{H(v_1,..,v_g)} \cong \mathbb{Z}^g$$

where $H(v_1,..,v_g)$ is the subgroup of $H_1(\partial X_1)$ generated by the homology classes of $\{v_1,..,v_g\}$ in ∂X_1.

ii) v_i is sent by the gluing map : $\partial X_1 \to \partial X_2$ into a trivial loop in X_2.

In this chapter we will prove that this manifold is determined uniquely (up to homeomorphism) if there is given a family of loops $\{w_1,..,w_k\}$ which satisfies the conditions i) and ii), even though $\{w_1,..,w_k\}$ are not necessarily simple or disjoint.

§2. Theorem.

The precise statement of our result is:

Theorem.

Let (X_i, X_i') be a Heegaard splitting of M_i $(i = 1,2)$, (i.e. $M_i = X_i \cup X_i'$, $X_i \cap X_i' = \partial X_i$, X_i and X_i' are handlebodies.)

Let $\{w_1,\ldots,w_k\}$ be a family of loops in ∂X_1 satisfying conditions i) and ii), and suppose there is a homeomorphism $g: X_1 \to X_2$ such that $g(w_i) \simeq 1$ in X_2'. Then M_1 is homeomorphic to M_2.

This theorem will follow from the lemma.

Lemma 1.

Let $i,j : S \to X$ be two embeddings of a closed surface S into the boundary of a handlebody X. If there exists a family of loops $\{w_1,\ldots,w_k\}$ in S such that

i) $(H_1(S)/H(w_1,\ldots,w_k)) \cong \mathbb{Z}^g$ $(g = \text{genus } X)$

ii) $\{w_1,\ldots,w_k\} \subset (\text{Kernel of } i_* : \pi(S) \to \pi(X)) \cap (\text{Kernel of } j_* : \pi(S) \to \pi(X))$.

Then $\mathrm{Ker}_{i_*} = \mathrm{Ker}_{j_*}$.

There is an algebraic version of this lemma (see §3).

Proof.

Let M be the manifold obtained by gluing two copies of X (say X and X') by the homeomorphism $i\, o(j|_{\partial X})^{-1} : \partial X \to \partial X'$. We will consider S as embedded in M and i (j) as the inclusion map of S into X (X').

If $i_{\#}$ $(j_{\#})$ is the induced homomorphism of i (j) in homology, then by hypothesis we have

$$H(w_1,\ldots,w_k) = (\ker i_{\#}) = (\ker j_{\#}) .$$

Thus there is a family of loops $\{e_1,\ldots,e_g,f_1,\ldots,f_g\}$ in S with $\langle e_i,f_j\rangle = \delta_i^j$, $\langle e_i,e_j\rangle = \langle f_i,f_j\rangle = 0$ $1\le i,j\le g$. ($\langle\ ,\ \rangle$ means the intersection number) and each e_i represents an element of

$$\mathrm{Ker}_{i_*} \cap \mathrm{Ker}_{j_*} \qquad (i = 1,\ldots,g)$$

So we have:

1. $H_1(M) \cong \mathbb{Z}^g$ and by duality $H_2(M) \cong \mathbb{Z}^g$

and

2. The natural map : $\pi_2(M) \to H_2(M)$ is onto. This follows from the fact that for each e_i there exists a singular two sphere

in M which meets S along e_i. Since $\langle e_i, f_j \rangle = \delta_i^j$ these spheres represent a base of $H_2(M)$.

By the sphere theorem there exists an embedded two sphere S^2 in M which represents a non-trivial element of $H_2(M)$ (*), i.e. S^2 is a non-separating two sphere in M. Therefore,

$M \cong (S^1 \times S^2) \# M'$, and using Mayer-Viecton's sequence we can see that $H_2(M') \cong \mathbb{Z}^{g-1}$ and the natural map : $\pi_2(M') \to H_2(M')$ is onto.

By induction it will follow that:

$$M \cong [\#_g (S^1 \times S^2)] \# \check{M} .$$

Since genus $M = g$ and the genus of a 3-manifold is additive with respect to connected sum, we must have genus $\check{M} = 0$. Thus $M \cong \#_g (S^1 \times S^2)$.

<u>Claim.</u>

For any Heegaard splitting (X,X') of genus g of $M \cong \#_g(S^1 \times S^2)$ there exists a family of loops of $\{v_1, \dots, v_g\}$ in $S = \partial X = \partial X'$ which is a system of meridional loops of X and X'.

<u>Proof.</u>

If $g = 1$ then $M \cong S^1 \times S^2$ and our result is trivial.

(*) Here we use the version of the sphere Theorem in [2], chapter 4 and we use the Kernel of the natural map : $\pi_2(M) \longrightarrow H_2(M)$ as the left π_1-invariant subgroup of $\pi_2(M)$.

If $g > 1$ then M admits an essential separating 2-sphere S^2 embedded in M which by Lemma 3, p.2.10, can be supposed to meet S in a single essential separating loop. So $M \cong M_1 \# M_2$ where the connected sum is made using the sphere S. By applying the inductive hypothesis to each M_i, our claim will follow. //

By our claim we must have

$$\mathrm{Ker}_{i_*} = \mathrm{Ker}_{j_*}$$

//

§3. Remarks.

Remark 1.

An algebraic version of lemma 1 is:

Lemma 1'

Let G be a sub group of $\pi(S)$ with

$$\frac{H_1(\pi(S))}{H_1(G)} \cong Z^g \quad , \quad g = \text{genus of } S \ .$$

Suppose there exists a normal subgroup N of $\pi(S)$ such that $\frac{\pi(S)}{N}$ is a free group of rank g (say F_g) and if $G \subset N$. Then N

is unique

$$(H_1(G) \text{ is the image of } G \text{ in } \frac{\pi(S)}{[\pi(S),\pi(S)]}) .$$

This lemma is equivalent to lemma 1 because any surjective homomorphism $f:\pi(S) \to F_g$ could be obtained as a homeomorphism of S with ∂X, where X is a handlebody (see [3]).

Remark 2.

In general there is no homeomorphism $j : S \to \partial X$ such that $j(w_i) \simeq 1$ in X, $i = 1,\ldots k.$, for loops $w_1,\ldots,w_k$ in S, even when

$$\frac{H_1(S)}{H_1(w_1,\ldots,w_k)} \cong \mathbb{Z}^g \quad \text{with} \quad \langle w_i,w_j\rangle = 0 .$$

For example consider $w_1 = \alpha_1$, $w_2 = \alpha_2\beta_1\beta_2\beta_1^{-1}\beta_2^{-1}$ where $\{\alpha_1,\alpha_2,\beta_1,\beta_2\}$ is a standard base of S (genus $S = 2$). (i.e. α_i and β_i are simple loops in $S(i = 1,2)$ and they have only the base point in common and $\langle\alpha_1,\alpha_2\rangle = \langle\beta_1,\beta_2\rangle = 0$ $\langle\alpha_i,\beta_i\rangle = 1$.

To see this, note that if there exists such map, $j(\alpha_1)$ will bound a disc in X. By looking at the intersection of this disc with the homotopy of $j(w_2)$ to the constant map we get $j(\beta_2) \simeq j(\alpha_2\beta_2^{-1}) \simeq 1$ in X and then $j_*:\pi(S) \to \pi(X)$ does not map onto $\pi(X)$. //

(*) Take a homotopy of $j(w_2)$ to a constant map in X (say, $\psi : D \to X$ transverse to a proper embedded disc in X (say D_1), the boundary of which is $j(\alpha_1)$.

Since $\psi|_{\partial D}$ is homotopic to $j(w_2)$ in ∂X, we can suppose

$\Rightarrow$ cont....

that $\Psi|_{\partial D}$ meets ∂D_1 in only two crossing points, separating $\Psi(\partial D)$ into two arcs A_1 and A_2. Then there exists a unique proper arc L in $\Psi^{-1}(D_1) \subset D$. $\Psi(L)$ is homotopic in D_1 (rel (∂L)) to an arc $A \subset D_1$. Therefore the loops $A_1 \cup A$ and $A_2 \cup A$ are trivial in X and they are free homotopic to $j(\beta_2)$ and $j(\alpha_2 \beta_2^{-1})$ in ∂X (not necessarily in this order).

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REFERENCES.

1. W. HAKEN: Some results on surfaces in 3-manifolds: M.A.A. Studies in Mathematics, Vol. 5; 1968
2. J. HEMPEL: Three-manifolds, Annals of Mathematic Studies No.86, Princeton, Princeton University Press, 1976;
3. W. JACO: Heegaard splitting and splitting homeomorphisms, Trans. Amer. Math. Soc. 144, 365-379 (1969);
4. E.E. MOISE: Geometric topology in dimensions 2 and 3, Graduate texts in Mathematics, Vol. 47, Springer Verlag, 1977;
5. C.D. PAPKYRIAKOPOULOS: On Dehn's lemma and the asphericity of knots. Ann. Math. 66. 1-26 (1957);
6. H. SEIFERT and W. THRELFALL: Lehnbuch der Topologie, Teubuer, Leipzig 1934;
7. F. WALDHAUSEN: Heegard-Zerlegungen der 3-Sphäre, Topology 7, 195-203 (1968).